Determinants of Special Circulant Matrices

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Introduction

A circulant matrix is a specific kind of Toeplitz matrix where each row vector is the right-circular shift of the preceding row. Circulant matrices appear naturally in a variety of applications including cryptography, digital signal processing, and image processing and compression.

In this project, some properties of circulant matrices with respect to their determinants were explored. In particular, a special type of sparse symmetric circulant matrix with each row containing three integer entries and the rest zeros was closely studied. The determinants of these matrices of various sizes were computed and analyzed using SageMath, an open-source CAS based on Python.

A few formulas were conjectured by observing the patterns related to the determinants involving divisibility properties. These conjectures were numerically verified up to certain matrix sizes due to computational limitations.

Definitions & Known Properties

1 Circulant matrix is defined as

$$C_n(c_0, c_1, \dots, c_{n-1}) = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{bmatrix}.$$

2. $Det\ C_n(c_0, c_1, \dots, c_{n-1}) = \prod_{k=0}^{n-1} \sum_{j=0}^{n-1} c_j \omega^{kj}$, where $\omega = e^{\frac{2\pi i}{n}}$.

3. Denote $D_n(d_1, d_2, d_3) = \begin{bmatrix} d_2 & d_3 & 0 & \cdots & d_1 \\ d_1 & d_2 & d_3 & \cdots & 0 \\ 0 & d_1 & d_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_3 & 0 & 0 & \cdots & d_2 \end{bmatrix}$, for $n \geq 3$.

Defining the circulant matrix D_n in SageMath:

```
def D(n,d1,d2,d3):
    M = np.zeros((n,n))
    M[0,0]=d2;M[0,1]=d3;M[0,n-1]=d1
    for i in [1..(n-1)]:
        for j in [0..(n-1)]:
            M[i,j]=M[i-1,j-1]
    return M
```

Conjecture 1

Let $W_n(k) = D_n(k,k,k)$ for $k \in \mathbb{N}$.

1. $\det(W_n(k))=0$ for $n = 3\ell$, where $\ell \in \mathbb{N}$.
2. $\det(W_n(1)) = \begin{cases} 0, & n \equiv 0 \text{ or } 3 (\text{mod } 6) \\ 3, & n \equiv 1 \text{ or } 5 (\text{mod } 6) \\ -3, & n \equiv 2 \text{ or } 4 (\text{mod } 6). \end{cases}$

Conjecture 2

Let $M_n(k) = D_n(k, k+2, k)$ for $k \in \mathbb{N}$ and $a_{n,k} = \sqrt{\frac{|det(M_n(k))|}{3k+2}}$.

Then, we have the following:

1. $a_{n,3}$ is a perfect square.
2. If k = 3 or 6, then $a_{n,k}$ divides $a_{n\ell,k}$, for all $\ell \in \mathbb{N}$.
3. If k=-3,-4,-5,-6, 4, 5 and
   a. n is odd, then
      i. $a_{n,k} \in \mathbb{Z}$ and
      ii. $a_{n,k}$ divides $a_{n(2\ell+1),k}$ for all $\ell \in \mathbb{Z}$.
   b. n is even, then
      i. $a_{n,k} \notin \mathbb{Q}$ and
      ii. $\frac{a_{n\ell,k}}{a_{n,k}} \in \mathbb{Z}$.

Values of $\frac{a_{n\ell,k}}{a_{n,k}}$ for $k = 4$

n	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
3	1	X	136	X
4	1	4	240	1984
5	1	X	2848	X
6	1	72	1088	216576
7	1	X	5248	X
8	1	496	180480	57012224
9	1	X	249344	X
10	1	1824	2278400	2243198976

X denotes that the computed value was not an integer.

Conjecture 3

$$\det(M_n(2)) = \begin{cases} 0, & n \text{ is even} \\ 2^{n+2}, & n \text{ is odd.} \end{cases}$$

Values of $\det(M_n(2))$

n	3	4	5	6	7	...	42	43
det	32.0	0	128.0	0	512.0	...	0	2^{45}

Conjecture 4

Let $V_n = D_n(1,2,1)$.

$$\det(V_n) = \begin{cases} 0, & n \text{ is even} \\ 4, & n \text{ is odd.} \end{cases}$$

```
d1=1;d2=2;d3=1;
for n in [3..50]:
    print(n,"\t",np.linalg.det(D(n,d1,d2,d3)).round())
```

Values of $\det(V_n)$

n	3	4	5	6	7	...	49	50
det	4.0	0	4.0	0	4.0	...	4.0	0

Remarks & Future work

All of these conjectures were verified with matrix sizes of at least 43. Due the computational limitations of the tools used in this project, determinants of larger matrices have rounding errors.

A continuation of this project will involve proving these conjectures and identifying some applications.

References

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